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A novel geomagnetic satellite constellation: Science and applications

Keke Zhang*

State Key Laboratory of Lunar and Planetary Sciences, Macau University of Science and Technology; CNSA Macau Center for Space Exploration and Science, Macao, China

Key Points:

- Discussing a new novel constellation of the geomagnetic survey satellites and its scientific and practical applications.
- Using strongly eccentric orbits with perigee about 200 km.
- Mapping the Earth's three-dimensional magnetic field in unprecedented accuracies and details.

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Abstract: The European Space Agency (ESA)'s Swarm constellation of a trio of geomagnetic survey satellites in nearly circular polar orbits at altitude about 500 km was launched on 22 November 2013 and has been mapping the Earth's global magnetic field in unprecedented details, helping scientists better understand how the geomagnetic field is generated and maintained inside the Earth's fluid core and how the Earth's external magnetic environment is changing. This review discusses a new novel constellation of the geomagnetic survey satellites that consists of at least four satellites: two satellites are in lower-latitude and nearly circular orbits at altitude about 450 km; two further satellites are marked by nearly polar but strongly eccentric orbits with perigee about 200 km and apogee about 5000 km. The new geomagnetic satellites are equipped with highly stable optical benches, high-precision fluxgate magnetometers and scalar magnetometers which are capable of mapping the Earth's three-dimensional magnetic field in unprecedented accuracies and details. The new constellation will help elucidate different contributions to the measured geomagnetic field: the core dynamo field, the lithospheric magnetic field, the magnetic fields produced by currents in the ionosphere and the magnetosphere as well as by the currents coupling the ionosphere and magnetosphere, and the magnetic fields induced from the electrically conducting mantle, lithosphere and oceans. In comparison to the Swarm mission, it will provide higher-accuracy, higher-resolution and higher-dimension measurements of the geomagnetic field required for shedding new insights into the core dynamo processes and the Earth's space magnetic systems along with a wide range of important applications.

Keywords: core dynamo; geomagnetic field; mantle; ocean; ionosphere; magnetosphere

1. Introduction

It is widely accepted that the Earth's deep interior comprises a solid inner core with the geocentric radius of about 1200 km and a fluid outer core from the inner core boundary to about 3400 km, where a sufficiently strong heat flux across the outer core of iron-nickel composition (electrically conducting) maintains its liquid state while convection-driven or/and precession-driven flow generates the internal geomagnetic field via the geodynamo processes (Bullard, 1949; Zhang K and Busse, 1989; Hollerbach and Jones, 1993; Zhang K and Schubert, 2000; Christensen et al., 2001; Roberts and King, 2013; Davies et al., 2022). We shall refer to this part of the geomagnetic field measured in the exterior of the Earth as the core dynamo field in this review. The intensity of the core dynamo field varies from nearly 2.5×10^4 nano-Testla (nT) in the South Atlantic region to nearly 6.5×10^4 nT in the polar regions (Siberia and Antarctica) at the Earth's surface (Langel et al., 1980; Olsen et al., 2014; Olsen et al., 2015; Finlay et al., 2015). Whereas the Earth's magnetic field has witnessed the evolution of the Earth's history (Buffett et al., 1996; Labrosse, 2003), its interaction and interplay with the Earth's external magnetic environment has influenced its atmospheric processes together with its long-term change of climate.

The geomagnetic field measured in the exterior of the Earth is highly complex and made up of contributions from at least the following five sources (Sabaka et al., 2004; Sabaka et al., 2015; Olsen et al., 2016): (i) electric currents generated by the core dynamo operating in its liquid outer core; (ii) electric currents induced in the Earth's electrically conducting mantle, lithosphere and oceans; (iii) the magnetization of the Earth's crust; (iv) electric currents in the Earth's ionosphere and magnetosphere; and (v) electric currents coupling the ionosphere and the magnetosphere. It follows that, provided a sophisticated separation and determination of different sources of the measured geomagnetic field is

Correspondence to: K. Zhang, kzhang@ex.ac.uk
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made, the high-precision measurements of the geomagnetic field can be employed to probe the properties and processes of the Earth's core, mantle, lithosphere, oceans and external space magnetic environments (Olsen et al., 2014; Finlay et al., 2017). In other words, the Earth's magnetic field represents an essential and a unique link of the highly complicated jigsaw puzzle of a complete picture of the whole Earth system from its outer liquid core to its external plasma space (Tarduno et al., 2010).

In addition to its fundamental importance in Earth's sciences, the high-precision three-dimensional structure of the Earth's magnetic field and its spatial–temporal variation are also vital to many industrial and technological applications such as aviation, navigation, telecommunication, ubiquitous mobile phones, power grids, rail networks, the exploration of natural resources and satellite operations. Furthermore, there exist causal links between the Earth's magnetic field and the Earth's biosphere: the core dynamo field is regarded as a shield protecting the human habitats on the Earth against lethal influences from the solar and cosmic radiations (Luntama, 2017).

The Earth's magnetic field represents one of the most important geophysical phenomena that has been utilized and studied by human beings for a very long time. A history of the geomagnetic-field utilization and study is primarily marked by the following ten milestones. (i) It was more than one thousand years ago that Chinese invented the first magnetic compass and brought the Earth's magnetic field into the service of human beings. They also discovered the magnetic declination (the directional angle between true south and geomagnetic south). (ii) In his renowned book "De Magnete" published in 1600, Gilbert recognized the geomagnetic field as an important physical property of the Earth (Gilbert, 1600). (iii) In 1635 Gellibrand discovered the secular variation of the Earth's magnetic field (Gellibrand, 1635), which represents a revolutionary finding, by identifying the slow change of compass direction in London. (iv) In 1701 after surveying the Atlantic Ocean between 1698 and 1700, Halley published a map showing lines of equal magnetic declination for the Atlantic Ocean (Courtillot and Le Mouel, 2007), providing the first contour charts of magnetic declination that was vital to global navigation. (v) Around 1722 Graham noticed the daily variation of Earth's magnetic field (Graham, 1724) and, thus, unveiled the connection of geomagnetic field fluctuations with the aurora and solar winds. (vi) In 1839, Gauss demonstrated that the potential of the magnetic field in the exterior of the Earth can be readily represented by a spherical harmonic series and its gradient gives rise to the vector of the geomagnetic field, suggesting that the main magnetic field of the Earth was originated mainly in its deep interior. (vii) In 1873, Maxwell provided a unified mathematical description, now known as Maxwell's equations, of electromagnetism which forms the theoretical basis of understanding the Earth's magnetic field. (viii) In 1906, Brunhes discovered the geomagnetic reversals by finding the direction of the magnetic field in a basaltic lava flow and the underlying backed clays is opposite to that of the present Earth's magnetic field and, then, correctly interpreting that the opposite direction is as a consequence of the reverse of the Earth's magnetic field in geological history (Brunhes, 1906). (ix) In 1963, Vine and Matthews hypothesized that, when an ocean crust forms at a mid-Atlantic ridge, the crust cooling below the Curie temperature becomes magnetized in the direction of the Earth's magnetic field: twinned magnetic strips resulting from seafloor spreading on either side of the ridge preserve the signal of changes in geomagnetic polarity through time (Vine and Matthews, 1963). (x) Over the past decades, a powerful combination of geomagnetic field data from magnetic survey satellites, such as Orsted (launched in 1999), CHAMP (launched in 2000) and Swarm (launched in 2013) along with ground-based magnetic observatories, has led to much better understandings of the Earth's core flow, secular variation and secular acceleration of the core dynamo field (Olsen et al., 2015; Finlay et al., 2016), based on which a short-term evolution of the Earth's magnetic field may be forecast (Fournier et al., 2010; Aubert, 2015). It has been convincingly demonstrated that geomagnetic satellites together with marine, airborne and observatory magnetic data are able to provide high accuracy and complete global coverage of the Earth's magnetic field.

This review describes a new novel geomagnetic constellation that can offer a higher-precision and three-dimensional survey of the Earth's magnetic field alongside its external three-dimensional magnetic environment, and discusses key scientific research and applications in connection with the higher-precision and three-dimensional geomagnetic measurements. It is anticipated that the new constellation of geomagnetic survey satellites is capable of making significant contribution to mapping the core dynamo field and lithospheric magnetic fields with a finer resolution and accuracy; constructing the finer three-dimensional profile of the Earth's mantle conductivity; enabling the three-dimensional construction of upper ionospheric and lower magnetospheric characteristics via in situ measurements of three-dimensional magnetic fields and other relevant physical parameters; understanding the primary state of Sun–Earth interaction and space weather conditions; developing highly accurate models of the Earth's magnetic field for science and applications; discovering new natural phenomena in the Earth's magnetic system; and validating or verifying new geomagnetic theories.

The key scientific payloads of the geomagnetic satellites and their detailed descriptions are discussed in various papers of this special issue. In what follows we shall first provide a brief introduction to the new constellation of geomagnetic survey satellites in Section 2, followed by discussing the science and applications of the high-precision geomagnetic data expected from the constellation in Section 3. A summary and some concluding remarks are given in Section 4.

2. A New Constellation of Geomagnetic Survey Satellites

The new constellation of geomagnetic survey satellites consists of at least four satellites: the first is in lower-latitude orbits with inclination to the equatorial plane about 40 degree, expects to be launched at the beginning of 2023, and equipped with a highly stable optical bench (Zhang H, 2023) and a high-precision fluxgate magnetometer (which is capable of accurately measuring the vector of the Earth's magnetic field via the ferromagnetic properties of some materials, but with a long-term drift caused by changes in the electronics and coil system) and a high-precision scalar magnetometer (which is capable of accurately measuring the modulus of the field via quantum mechanical properties of liquid or gas without having long-term drifts); the second is also in lower-latitude orbits with inclination to the equatorial plane about 40 degree and to be launched together the first satellite, but its main objective is to monitor the solar activities and their effect on the Earth's space plasma; the third is marked by polar and strongly eccentric orbits with perigee about 200 km and apogee about 5000 km at an inclination nearly 90 degree, equipped with an optical bench and a fluxgate magnetometer and a scalar magnetometer, and is in the stage of planning; and the fourth is also marked by polar and strongly eccentric orbits with perigee about 200 km and apogee about 5000 km at an inclination nearly 90 degree but with a different orbital plane relative to the third, equipped with an optical bench, a fluxgate magnetometer and a scalar magnetometer, and expects to be lunched together with the third satellite. Note that, because of non-spherical symmetry of the Earth's gravitational field, the location of the satellite's perigee in an eccentric orbit drifts periodically from north pole to south pole, covering the whole Earth. It is anticipated that the new constellation will be able to provide, compared to the ESA's Swarm mission, higher-precision, higher-resolution, higher-dimension measurements of the Earth's magnetic field required to study various aspects of the geomagnetism (Jiang Y, 2023).

There exist nearly 400-year (or more) data of historical geomagnetic observations, about 200-year data of geomagnetic observatory measurements and more than 20-year data of continuous satellite measurements of the geomagnetic field. It should be underlined that the new constellation of geomagnetic survey satellites, characterized by a combination of nearly circular low-latitude and highly eccentric polar orbits, can provide not only unprecedentedly accurate and three-dimensional measurements that sample the near-Earth magnetic fields but also some key physical characteristics of the upper ionosphere and the lower magnetosphere continuously at different distances in the range of 200−5,000 km from the Earth's surface, leading to a better understanding of the geomagnetic field as well as the physics and dynamics of the ionosphere and magnetosphere.

It is expected that the new constellation of geomagnetic survey satellites would offer the following research platforms in geomagnetism: (i) providing excellent global and local time sampling of the Earth's magnetic field; (ii) leading to a three-dimensional structure of the near Earth's magnetic environment through the highly eccentric orbits; (iii) constraining the electrical conductivity profile of the Earth's mantle; (iv) improving the global model of lithospheric magnetic fields via the new faithful measurements in the locations of near 200 km perigee; (v) determining the secular variation of the core dynamo field with higher resolution; (vi) inverting and constraining fluid motion in the Earth's liquid outer core as well as forecasting the short-term variation of the core dynamo field; (vii) identifying the magnetic signatures produced by large-scale oceans flow and electric currents; and (viii) constructing nearly real time-dependent geomagnetic field models that include all significant sources with a wide range of timescales and spatial-scales and that have important scientific and industrial applications.

3. Science and Applications

3.1 Forecasting Core Dynamo Magnetic Field

The real Earth does not rotate uniformly with a constant angular velocity, the container of the core fluid, the core–mantle interface, is not perfectly spherical with a uniform boundary, and the fluid in the outer core is not completely incompressible. However, it is widely believed that these details are of secondary importance, *i.e.*, to a first approximation, they do not influence the primary dynamics, such as thermal convection, of the Earth's outer core in the forward problem (Gubbins and Roberts, 1987; Hollerbach and Jones, 1993; Zhang K and Gubbins, 2000). Similarly, the corresponding inverse problem—deriving fluid motion in the Earth's core from the three-dimensional magnetic field measured by geomagnetic survey satellites—usually does not consider the secondary aspects such as the non-spherical and non-uniformly-rotating effects of the real Earth (Whaler and Beggan, 2015). This review is only concerned with spherical geometry and uniform rotation of the Earth.

We consider the Earth's core dynamo as a convection-driven magnetohydrodynamic dynamo in a spherical shell of inner radius about $r_{\rm i}$ = 1200 km and outer radius about $r_{\rm o}$ = 3400 km filled with an electrically conducting, incompressible Boussinesq fluid which rotates with a uniform angular velocity $\boldsymbol{\Omega} = \Omega\hat{\boldsymbol{z}}$, where Ω is the angular speed of rotation assuming to be constant, and $\hat{\boldsymbol{z}}$ denotes the unit vector parallel to the axis of Earth's rotation. Gravity is assumed to varies linearly with radius, $\boldsymbol{g} = g_0\boldsymbol{r}$, where g_0 is constant and $\boldsymbol{r}$ is the position vector (Chandrasekhar, 1961). The outer-core fluid is assumed to have constant magnetic diffusivity η, magnetic permeability μ, kinematic viscosity ν, density ρ, thermal diffusivity κ and thermal expansion coefficient α. In the inner sphere $0 \le r \le r_{\rm i}$, we assume that there is a solid electrically conducting core with uniform magnetic diffusivity $\eta_{\rm i}$ in which the magnetic fields cannot be generated.

In the mantle frame of reference, the core dynamo magnetic field is described by Ohm's law and Maxwell's equations, which are, respectively,

$$\boldsymbol{J} = \frac{1}{\mu\eta}(\boldsymbol{u}\times\boldsymbol{B}+\boldsymbol{E}), \tag{1}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{2}$$

$$\boldsymbol{J} = \frac{1}{\mu}\nabla\times\boldsymbol{B}, \tag{3}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} = -\nabla\times\boldsymbol{E}, \tag{4}$$

where $\boldsymbol{J}$ represents the electric current, $\boldsymbol{u} = (u_r, u_\theta, u_\phi)$ is the three-dimensional velocity field, in spherical polar coordinates (r, θ, ϕ) with $\theta = 0$ being at the axis of rotation, $\boldsymbol{B} = (B_r, B_\theta, B_\phi)$ is the three-dimensional magnetic field, $\boldsymbol{E}$ is the electric field, and the electric conductivity $\sigma = 1/(\mu\eta)$. Note that the magnetohydrodynamic approximation is made for the non-relativistic flow (the flow's speed is much smaller than the speed of light) in the magnetohydrodynamics limit by neglecting the displacement current. Equations (1)–(4), together with the condition of incompressibility,

$$\nabla\cdot\boldsymbol{u} = 0 \tag{5}$$

can be conveniently combined together to give a single equation for the magnetic field $\boldsymbol{B}$, the magnetic induction equation or the dynamo equation, either in the form

$$\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{B} = \boldsymbol{B}\cdot\nabla\boldsymbol{u} + \eta\nabla^2\boldsymbol{B}, \tag{6}$$

or in the form

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) - \eta\nabla\times\nabla\times\boldsymbol{B}. \tag{7}$$

The dynamo Equation (6) or (7) can be utilized to invert the core flow $\boldsymbol{u}$ from the observed magnetic field $\boldsymbol{B}$ and its secular variation $\partial\boldsymbol{B}/\partial t$.

In addition to the dynamo equation, the core dynamo in $r_{\rm i} \le r \le r_{\rm o}$ is governed by the equations of motion and energy respectively,

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + 2\boldsymbol{\Omega}\times\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \frac{\alpha g_o\boldsymbol{r}}{r_o}\Theta + \nu\nabla^2\boldsymbol{u} + \frac{1}{\rho\mu}(\nabla\times\boldsymbol{B})\times\boldsymbol{B}, \tag{8}$$

$$\frac{\partial\Theta}{\partial t} + \boldsymbol{u}\cdot\nabla\Theta = -\boldsymbol{u}\cdot\nabla T_0(r) + \kappa\nabla^2\Theta, \tag{9}$$

where $p = p(\boldsymbol{r}, t)$ denotes the pressure and $\Theta(\boldsymbol{r}, t)$ represents the temperature departure from the basic conducting temperature $T_0(r)$. We may assume, for the purpose of simplicity, that the spherical-shell outer core is heated from below to maintain a higher temperature $\hat{T}_{\rm i}$ at $r = r_{\rm i}$ and a lower temperature $\hat{T}_{\rm o}$ at $r = r_{\rm o}$. It is required that $(\hat{T}_{\rm i} - \hat{T}_{\rm o})$, or the gradient of $T_0(r)$, must be sufficiently large to produce the unstable temperature profile $T_0(r)$ to drive thermal convection (Zhang K, 1992) and generate the convection-driven, core-dynamo field $\boldsymbol{B}$ via the magnetohydrodynamic processes (Gubbins and Roberts, 1987; Zhang K and Schubert, 2000).

The core dynamo problem involves solving the nonlinear system governed by Equations (2), (5) and (7)–(9) together with appropriate boundary conditions for $\boldsymbol{u}$, $\boldsymbol{B}$ and Θ (Zhang K and Busse, 1989; Hollerbach and Jones, 1993). Two different types of velocity boundary condition are widely adopted: non-slip and impenetrable, *i.e.*,

$$u_\theta = u_\phi = u_r = 0, \tag{10}$$

on the inner and outer bounding spherical surfaces, or stress-free and impenetrable, *i.e.*,

$$\frac{\partial(u_\phi/r)}{\partial r} = \frac{\partial(u_\theta/r)}{\partial r} = u_r = 0. \tag{11}$$

Boundary condition (10) produces strong viscous boundary layers at the bounding surfaces of the outer core due to the effect of the Earth's rapid rotation, while Equation (11) gives rise to much weaker viscous boundary layers.

There exist also two different types of the magnetic boundary condition which are widely employed: a perfectly electrically insulating exterior and an exterior that has the same electrical conductivity as that of the fluid. In both the cases, the magnetic field $\boldsymbol{B}$ must be continuous across the interfaces between the electrically conducting fluid and the exterior,

$$[\boldsymbol{B} - \boldsymbol{B}^e] = \boldsymbol{0}, \tag{12}$$

on the inner and outer bounding surfaces of the outer core, where $\boldsymbol{B}^e$ denotes the magnetic field in the exterior. Moreover, the magnetic field must also decay like

$$|\boldsymbol{B}| \sim O(r^{-3}), \quad \text{as} \quad r\to\infty. \tag{13}$$

Since the temperature boundary condition usually does not play a major role, the most commonly used ones are either perfectly thermally conducting,

$$\Theta = 0, \tag{14}$$

or a constant heat flux that is maintained at all times

$$\frac{\partial\Theta}{\partial r} = 0 \tag{15}$$

on the inner or outer bounding surfaces of the fluid core.

Upon adopting an appropriate scaling, the core dynamo problem is characterized by four independent dimensionless parameters, regardless of the particular scalings adopted. The four frequently-used parameters are usually the Rayleigh number Ra, the Ekman number Ek, the Prandtl number Pr and the magnetic Prandtl number Pm,

$$Ra \equiv \frac{\alpha\beta g_0 d^4}{\nu\kappa},\ Ek \equiv \frac{\nu}{2\Omega d^2},\ Pr \equiv \frac{\nu}{\kappa},\ Pm \equiv \frac{\nu}{\lambda}, \tag{16}$$

where d is the thickness of the liquid core $d = r_{\rm o} - r_{\rm i}$, and β is an average gradient of the basic temperature state T_0. Other parameters can be obtained from a combination of the four numbers. For example, the Taylor number Ta is related to the Ekman number Ek by $Ta = Ek^{-2}$. The Rayleigh number Ra provides a measure of the strength of applied buoyancy forces and is associated with the superadiabatic temperature gradient in the Earth's liquid core which cannot be precisely determined.

There are three different diffusive processes—magnetic, thermal and viscous—which take place in the Earth's core dynamo. The values of the Prandtl number Pr and the magnetic Prandtl number Pm indicate the relative importance of these three diffusive processes that are determined by the material properties of an electrically conducting fluid. The most important parameter, as far as the dynamics of the Earth's liquid core is concerned, is the Ekman number Ek which is extremely small. For example, a typical value of the Ekman number Ek for the Earth's fluid core is as small as $Ek = 10^{-15}$ (Gubbins and Roberts, 1987). It is this extremely small value of Ek that makes numerical simulations or forecasts of the core dynamo not only fascinating but also insurmountably difficult (Zhang K and Jones, 1997).

No core dynamo simulations have been performed with geophysically realistic parameters and are capable of describing the spatial and temporal scales of geomagnetic observations. Constructing a core dynamo model within the Earth's parameter regime, even though some existing core dynamo models with unrealistic geophysical parameters exhibit Earth-like behaviors, is beyond the current computing power and remains to be a grand challenge in the foreseeable future. It is hoped, however, that numerical simulations of Equations (2), (5), (7)–(9) together an appropriate data assimilation algorithm (Fournier et al., 2010; Aubert, 2015; Sanchez et al., 2020) can be performed at sufficiently small values of Ek so as to capture the correct dynamics for an asymptotically small Ek and, thus, permit an extrapolation of the physical behavior of the core dynamo to an asymptotically realistic but numerically unreachable small Ek. The future progress on forecasting the variation of the core dynamo magnetic field, particularly for a relatively short term, depends on (i) a successful application of the modern Artificial Intelligence (AI) technology to the geodynamo data assimilation; (ii) a joint analysis of the geomagnetic constellation measurements and the ground/airborne observations to provide better constraints in the models; (iii) a combined analysis of accurate global geomagnetic measurements with the core dynamo data assimilation; and (iv) a reliable separation of the core dynamo field from other contributions such as induced magnetic fields from the mantle and oceans.

With high-precision measurements from the new geomagnetic constellation, the dynamo data assimilation can be regarded as a strong regularization in which the forecasted magnetic field of different geomagnetic models at different parameter regimes trail an Ensemble of the core dynamo simulations governed by Equations (2), (5) and (7)–(9), driven by thermal convection in the Earth's outer core, accessible to modern supercomputer powers, and being able to provide a reasonably accurate forecasting of the short-term variation of the core dynamo field of the Earth (Li JF et al., 2023).

3.2 Determining Fluid Motion in the Earth's Core

Under the most extreme condition within the whole Earth, fluid motion in its outer core is completely inaccessible though it offers key insight into the complicated dynamic process of the core dynamo. It is the measurements of the core dynamo field, mainly via geomagnetic survey satellites, that may determine the fluid motion and, thus, reveal a wealth of dynamic phenomena in the outer core (Gubbins, 1982; Le Mouel, 1984; Bloxham and Jackson, 1991; Holme and Olsen, 2006). For example, Finlay and Jackson (2003) revealed an interesting phenomenon of the secular variation where the geomagnetic flux patches in the equatorial region, mainly associated with the azimuthal wavenumber $m = 5$, drift westward at a speed of approximately 17 km per year, suggesting a magnetically modified equatorially trapped inertial wave with $m = 5$ (Zhang K, 1993).

Extracting the useful information of mass motion in the Earth's fluid core, particularly beneath the core–mantle boundary, from the high-precision measurements of geomagnetic survey satellites represents a classical inverse problem that is marked by the inherent difficulties of non-uniqueness (Roberts and Scott, 1965). It implies that there exist many different patterns of fluid motion in the outer core that can give rise to the same pattern of geomagnetic measurements. To resolve the non-uniqueness, both high-quality geomagnetic data and theoretical advances associated with various hypotheses are required. Below are several widely adopted hypotheses used in determining fluid motion in the Earth's core:

(i) The Earth's mantle is assumed to be a perfectly electric insulator, *i.e.*, the electric current cannot enter the mantle from the core dynamo and the geomagnetic field in the mantle is simply governed by the equations

$$0 = \nabla \cdot \boldsymbol{B}, \tag{17}$$

$$0 = \nabla \times \boldsymbol{B}, \tag{18}$$

with the condition $\boldsymbol{r} \cdot \boldsymbol{J} = 0$ together with a continuous poloidal magnetic field at the core–mantle boundary. This allows us easily to compute or infer the geomagnetic field at the core–mantle boundary from the satellite measurements made in the exterior of the Earth. In other words, only the poloidal component—a solenoidal magnetic field satisfying Equation (17) can be always decomposed into the toroidal and poloidal components (Chandrasekhar, 1961)—of the magnetic field can be observed.

(ii) The Earth's outer core is assumed to be a perfectly electric conductor governed by the following induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}), \tag{19}$$

where the variation of the core dynamo field is caused only by magneto-advection (which has the time scale of less than $O(10^2)$ years) without having magnetic diffusion (which has the time scale of more than $O(10^4)$ years) (Gubbins and Roberts, 1987). If concerned timescales are much shorter than that of the magnetic diffusion and if length scales of the flow $\boldsymbol{u}$ and the field $\boldsymbol{B}$ are similar, the diffusive effects in connection with the term $\nabla^2 \boldsymbol{B}$ may be neglected, leading to the widely-used frozen flux approximation—the magnetic field lines in the perfectly conducting fluid core are frozen with the fluid motion (Bloxham and Jackson, 1991). Take the radial component of Equation (19) at the core–mantle boundary,

$$\frac{\partial B_r}{\partial t} = -\frac{1}{r\sin\theta}\left[\frac{\partial(u_\theta B_r)\sin\theta}{\partial\theta} + \frac{\partial(u_\phi B_r)}{\partial\phi}\right], \tag{20}$$

the resulting Equation (20) may be utilized to find the flow $[u_\theta(\theta, \phi, r = r_o), u_\phi(\theta, \phi, r = r_o)]$ via the observed geomagnetic field B_r and its secular variation $\partial B_r/\partial t$ at the core–mantle boundary. Evidently, there exists a profound non-uniqueness in this inverse problem: a single scalar Equation (20) is insufficient to solve for the two components of flow u_ϕ and u_θ at $r = r_o$. In other words, the observed core dynamo magnetic field, B_r and its variation $\partial B_r/\partial t$, from geomagnetic survey satellites cannot determine uniquely the flow at the core–mantle boundary (Roberts and Scott, 1965) and, thus, some additional assumptions are required.

(iii) The Earth's fluid core is assumed to be dynamically geostrophic and approximately described by

$$2\boldsymbol{\Omega} \times \boldsymbol{u} = -\frac{1}{\rho}\nabla p, \tag{21}$$

where both the viscous force $\nu\nabla^2 \boldsymbol{u}$ and the Lorentz force are neglected because of

$$\frac{|\nu\nabla^2 \boldsymbol{u}|}{|2\boldsymbol{\Omega} \times \boldsymbol{u}|} \ll 1$$

and

$$\frac{|(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}|}{\rho\mu|2\boldsymbol{\Omega} \times \boldsymbol{u}|} \ll 1.$$

It is also assumed that the invisible toroidal magnetic field (Zhang K, 1995) and its derivative are small at the core–mantle boundary $r = r_o$, consistent with the hypothesis that the mantle is a perfect insulator and, thus, the toroidal component of the magnetic field must vanish there. Curling Equation (21) and taking its radial component give rise to the geostrophic constraint

$$\frac{\partial(u_\theta(\theta,\phi)\sin\theta\cos\theta)}{\partial\theta}+\frac{\partial(u_\phi(\theta,\phi)\cos\theta)}{\partial\phi}=0, \tag{22}$$

at the core–mantle boundary, which can be employed to reduce the non-uniqueness of the inverse problem. It should be noticed, however, that the assumed vanishing viscosity $\nu = 0$ does not hold at the core–mantle boundary as it cannot satisfy the required velocity boundary condition and, consequently, a viscous boundary layer must be introduced there, which can dramatically complicate the problem (Zhang K et al., 2007).

(iv) The Earth's fluid core is assumed to be dynamically quasi-geostrophic and approximately described by

$$\frac{\partial \boldsymbol{u}}{\partial t}+2\boldsymbol{\Omega}\times\boldsymbol{u}+\frac{1}{\rho}\nabla p=0,\quad \nabla\cdot\boldsymbol{u}=0, \tag{23}$$

where the quasi-geostrophy means

$$0<\frac{|\partial\boldsymbol{u}/\partial t|}{|2\boldsymbol{\Omega}\times\boldsymbol{u}|}\ll 1,$$

which represents the slowly-changing leading-order flow of thermal convection in rapidly rotating spherical geometry. Note that the general analytical solution to Equation (23) is available in a full sphere geometry (Zhang K and Liao XH, 2017). A solution to Equation (23) can be therefore written in the form

$$\boldsymbol{u}(r,\theta,\phi,t)=\sum_{mnk}C_{mnk}(t)\boldsymbol{u}_{mnk}(r,\theta,\phi), \tag{24}$$

where $\boldsymbol{u}_{mnk}(r,\theta,\phi)$ denotes the quasi-geostrophic inertial modes. The unknown coefficients C_{mnk} can be determined via B_r and its variation $\partial B_r/\partial t$ provided by the measurements of geomagnetic survey satellites (Kloss and Finlay, 2019). Relaxing the geostrophic constraint Equation (21) to the quasi-geostrophic constraint Equation (23)—which allows a weak departure from the exact geostrophy—enables to make use of the prior dynamical information on the core convective flow to mitigate the non-uniqueness of the inverse problem and offer the fully three-dimensional information of the flow.

In summary, determining mass motion in Earth's fluid core requires the high-precision measurements of geomagnetic survey satellites together with a deep understanding of its magnetohydrodynamics to mitigate the inherent difficulties of non-uniqueness. The information of fluid motion in the Earth's core offers key insight into the core dynamo process and helps forecast the geomagnetic secular variations.

3.3 Probing the Electrical Conductivity of the Earth's Mantle

In the real Earth, its lithosphere and its mantle are not perfectly electrical insulators and, consequently, external changes in the Earth's magnetic field—which are primarily caused by the solar-wind interaction with the ionosphere and magnetosphere at periods much shorter than the eleven-year sunspot cycle—can induce electric currents in the Earth's lithosphere and mantle and, then, produce a secondary magnetic field of the amplitude $O(10)$ nT measurable by ground magnetic observatories and high-accuracy geomagnetic satellites (Banks, 1969; Kuvshinov, 2008). The profile and magnitude of the secondary magnetic field open up a means of probing three-dimensional conductivity and heterogeneity of the Earth's mantle that is closely connected with its physical and chemical properties, such as porosity, the amount and distribution of water, temperature and composition (Kelbert et al., 2009; Yoshino, 2010). Consequently, studies of the electromagnetic induction in the Earth's lithosphere and mantle of lateral heterogeneity complement those of seismic tomography that is mainly associated with the mechanical properties of the Earth's lithosphere and mantle.

The problem of the global electromagnetic induction in the Earth's lithosphere and mantle can be mathematically formulated via a perturbation approach by assuming that the ionospheric dynamo is decoupled from the core dynamo. As an example of illustration, below is a perturbation formulation focusing on the the secondary magnetic field induced by the Earth's ionospheric current that is maintained by the ionospheric dynamo.

First, consider the full induction problem in the mantle frame of reference described by Ampere's current law and Faraday's law of induction at a particular frequency ω in the magnetohydrodynamic limit,

$$\frac{1}{\mu}\nabla\times\boldsymbol{B}(r,\theta,\phi)=\sigma(r,\theta,\phi)\boldsymbol{E}(r,\theta,\phi), \tag{25}$$

$$\mathrm{i}\omega\boldsymbol{B}(r,\theta,\phi)=-\nabla\times\boldsymbol{E}(r,\theta,\phi), \tag{26}$$

$$\nabla\cdot\boldsymbol{B}(r,\theta,\phi)=0, \tag{27}$$

where $\mathrm{i}=\sqrt{-1}$. The problem is defined in the spherical domain $r_{\rm o}\le r\le r_{\rm source}$ in which $r_{\rm o}\le r\le r_{\rm E}$ denotes the mantle while $r_{\rm E}\le r\le r_{\rm source}$ denotes the exterior between the Earth's surface at $r=r_{\rm E}$ and the location of the extraneous electric currents at $r=r_{\rm source}$. Note that the electrical conductivity σ in the domain $r_{\rm o}\le r\le r_{\rm source}$ is also comprised of the two parts: (i) $\sigma=\sigma_m(r,\theta,\phi)$ in the mantle $r_{\rm o}\le r\le r_{\rm E}$, which is $O(1)$ S/m in the upper mantle and (ii) $\sigma=\sigma_a$ in the air $r_{\rm E}\le r\le r_{\rm source}$, which is small and may be assumed to be zero. A harmonic time dependence of the magnetic magnetic $\boldsymbol{B}$ and the electric field $\boldsymbol{E}$ is assumed to be of the form

$$\boldsymbol{B}=\boldsymbol{B}(r,\theta,\phi)\mathrm{e}^{\mathrm{i}\omega t},$$
$$\boldsymbol{E}=\boldsymbol{E}(r,\theta,\phi)\mathrm{e}^{\mathrm{i}\omega t}.$$

This is reasonable because the ionospheric dynamo, driven mainly by the solar tide, is nearly periodic. It follows that the full induction problem, a spherical boundary value problem, is governed by Equations (25)–(27) and subject to the two boundary conditions: at the core–mantle boundary $r=r_{\rm o}$, we impose that

$$\boldsymbol{B}=0\quad\text{at}\quad r=r_{\rm o}, \tag{28}$$

as the electrical conductivity of the fluid core, in contrast to the mantle, is nearly perfect; at the location of the extraneous electric current, we impose that

$$\boldsymbol{J}^{\rm ext}=\mathcal{F}_{\rm source}(\theta,\phi)\quad\text{at}\quad r=r_{\rm source}, \tag{29}$$

where $\mathcal{F}_{\rm source}(\theta,\phi)$ is regarded as a known function by assuming that the ionospheric dynamo is confined within a thin spherical layer at $r=r_{\rm source}$. Note that the skin depth, the penetrating distance in which the induced magnetic field amplitude decreases by a factor of $1/e$, is approximately give by $\sqrt{2/(\omega\sigma\mu)}$.

The full induction problem—defined by Equations (25)–(27) and subject to the conditions (28)–(29)—cannot separate the primary field $\boldsymbol{B}_1$ generated by the ionospheric dynamo from the secondary field $\boldsymbol{B}_2$ induced by the mantle. We thus decompose the total measured field $\boldsymbol{B}$ into the two parts: $\boldsymbol{B} = \boldsymbol{B}_1 + \boldsymbol{B}_2$ assuming that $|\boldsymbol{B}_1| \gg |\boldsymbol{B}_2|$. Similarly, the total electric field $\boldsymbol{E}$ is also decomposed into the two parts: $\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2$ assuming that $|\boldsymbol{E}_1| \gg |\boldsymbol{E}_2|$. Since the electrical conductivity $\sigma_{\rm m}$ in the mantle is relatively small, it is mathematically advantageous and physically meaningful to take a perturbation approach by first solving a homogeneous equation with inhomogeneous boundary conditions for the primary magnetic field $\boldsymbol{B}_1$ and, then, solving an inhomogeneous equation with homogeneous boundary conditions for the secondary magnetic field $\boldsymbol{B}_2$.

It follows that the leading-order problem is described by the homogeneous equations

$$\frac{1}{\mu}\nabla \times \boldsymbol{B}_1(r,\theta,\phi) = 0, \tag{30}$$

$$\mathrm{i}\omega\boldsymbol{B}_1(r,\theta,\phi) + \nabla \times \boldsymbol{E}_1(r,\theta,\phi) = 0, \tag{31}$$

$$\nabla \cdot \boldsymbol{B}_1(r,\theta,\phi) = 0, \tag{32}$$

in the whole domain $r_{\rm o} \le r \le r_{\rm source}$, subject to the two boundary conditions

$$\boldsymbol{B}_1 = \boldsymbol{0} \;\; \text{at} \;\; r = r_{\rm o}, \tag{33}$$

and

$$\boldsymbol{J}^{\rm ext} = \mathcal{F}_{\rm source}(\theta,\phi) \;\; \text{at} \;\; r = r_{\rm source}. \tag{34}$$

The leading-order solution provides the primary magnetic and electric field, $\boldsymbol{B}_1$ and $\boldsymbol{E}_1$, in the whole domain $r_{\rm o} \le r \le r_{\rm source}$ generated by the ionospheric dynamo.

The next-order problem in the mantle $r_{\rm o} \le r \le r_{\rm E}$ is then described by the inhomogeneous equations

$$\frac{1}{\mu}\nabla \times \boldsymbol{B}_2(r,\theta,\phi) = \sigma_{\rm m}\boldsymbol{E}_1(r,\theta,\phi), \tag{35}$$

$$\mathrm{i}\omega\boldsymbol{B}_2(r,\theta,\phi) + \nabla \times \boldsymbol{E}_2(r,\theta,\phi) = 0, \tag{36}$$

$$\nabla \cdot \boldsymbol{B}_2(r,\theta,\phi) = 0, \tag{37}$$

subject to the boundary condition at the core–mantle boundary

$$\boldsymbol{B}_2 = \boldsymbol{0} \;\; \text{at} \;\; r = r_{\rm o}, \tag{38}$$

as a result of the rapid attenuation of the induced field in the upper mantle. Note that the term $\sigma_{\rm m}\boldsymbol{E}_2(r,\theta,\phi)$, because of $|\boldsymbol{E}_2| \ll |\boldsymbol{E}_1|$, is neglected in Equation (35).

As clearly implied by Faraday's law of induction Equation (26), the temporal variation of the primary magnetic field $\partial\boldsymbol{B}_1/\partial t$, which is typically at periods much less than a year, induces the primary electric field $\boldsymbol{E}_1$ that drives the secondary electric current $\sigma_{\rm m}\boldsymbol{E}_1$ described by Ampere's law (35) and, then, generates the secondary magnetic field $\boldsymbol{B}_2$ measurable by the high-precision geomagnetic satellites. In the external domain $r_{\rm E} \le r \le r_{\rm source}$, the secondary magnetic field $\boldsymbol{B}_2$ can be written as the gradient of a magnetic scalar potential Φ_2,

$$\boldsymbol{B}_2 = -\nabla\Phi_2(r,\theta,\phi), \tag{39}$$

satisfying the Laplace's equation

$$\nabla^2\Phi_2(r,\theta,\phi) = 0. \tag{40}$$

It is coupled with the induced magnetic field at the Earth's surface through the marching condition

$$-\nabla\Phi_2(r_{\rm o},\theta,\phi) = \boldsymbol{B}_2 \;\; \text{at} \;\; r = r_{\rm E}. \tag{41}$$

Since the secondary magnetic field $\boldsymbol{B}_2$ induced by the Earth's mantle is heavily attenuated and becomes negligibly small at the location of the ionospheric dynamo, we may impose the following boundary condition,

$$\nabla\Phi_2 = 0 \;\; \text{at} \;\; r = r_{\rm source}. \tag{42}$$

The second-order solution in this perturbation approach provides the secondary magnetic field $\boldsymbol{B}_2$ in the whole domain $r_{\rm i} \le r \le r_{\rm source}$. Equations (35)–(37) show that the amplitude and structure of the secondary magnetic field $\boldsymbol{B}_2$ are largely dependent upon both the frequency ω and the lateral heterogeneity of the conductivity profile $\sigma_{\rm m}(r,\theta,\phi)$.

There is a forward problem as well as an inverse problem in the study of the electromagnetic induction for the Earth's lithosphere and mantle. In the forward problem, we predict both the primary magnetic field $\boldsymbol{B}_1$ and the secondary magnetic field $\boldsymbol{B}_2$ (Yao HB et al., 2021) for a given electrical conductivity $\sigma_{\rm m}(r,\theta,\phi)$, a given frequency ω and a source boundary condition (29). In the inverse problem, we estimate the electrical conductivity $\sigma_{\rm m}(r,\theta,\phi)$ (Kuvshinov and and Olsen, 2006; Kuvshinov and Semenov, 2012; Püthe et al., 2015; Yao HB et al., 2022) by solving an optimization problem which minimizes a penalty function that involves the observed primary magnetic field $\boldsymbol{B}_1$ and the observed secondary magnetic field $\boldsymbol{B}_2$, the forward problem, the magnetic field data misfit, the parameterized electrical conductivity, and the regularization parameter and the regularization term—which is characterized by an inherent non-uniqueness of the inverse problem.

In summary, dynamic processes in the Earth's mantle—mantle convection and plate tectonics—generate seismic waves that are widely used to infer the dynamic state of the mantle while the secondary magnetic field $\boldsymbol{B}_2$, measured by geomagnetic observatories and high-accuracy geomagnetic satellites, opens up the possibility of probing the three-dimensional conductivity and heterogeneity in the Earth's lithosphere and mantle. A powerful combination of the seismological and geomagnetic studies of the Earth's mantle, together with constraints from mineral physics, geochemistry, geodynamics and magnetohydrodynamics, contribute to our better understanding of the dynamics and evolution of the deep Earth, offering a unique constraint between geomagnetism and seismic tomography on the physical and chemical heterogeneity in the upper mantle that is unattainable alone via seismological studies.

3.4 Mapping Magnetic Field of the Earth's Lithosphere

After the Earth's hot magma emerges from volcanoes and when the ambient temperature deceases below the Curie temperature, crustal magnetization takes place with the amplitude and direction of the core dynamo field being frozen in the cooling rock at that instant of geologic time (Backus et al., 1996; Langel and Hinze, 1998). It is usually assumed that the interface between the Earth's crustal magnetic sources and the mantle is marked by the location of the Curie temperature at which crustal rocks lose their magnetization. Magnetic signatures of the Earth's lithosphere recorded in rock magnetization can thus unveil the slow movement of Earth's continents and are usually employed in interpreting and understanding large-scale geological processes such as plate tectonics. Profiles of lithospheric magnetization $\boldsymbol{M}(r,\theta,\phi)$ have played an essential role in untangling the history of the Earth's tectonics in various geological epochs, offering a unique magnetic fingerprint deployed to define and study polar wonder, seafloor spreading and plate movement (Molnar, 1988).

Earth's lithospheric magnetization $\boldsymbol{M}(r,\theta,\phi)$ is of two different types: remanent and induced. The remanent magnetization persists even after the core dynamo has demised while the induced magnetization is proportional to the amplitude and direction of the inducing core dynamo field and vanishes when the inducing magnetic field vanishes. Earth's crustal magnetization in its continents is primarily induced while lithospheric magnetization in the ridge region of oceans is mainly remanent.

In the temporal spectrum, the lithospheric magnetic field, in comparison with the magnetic fields originating from other sources, is simply regarded as being stationary and permanent. There exists, however, a broad spatial spectrum of the lithospheric magnetic field, ranging from $O(10)$ km to $O(1000)$ km, produced by the magnetized rocks in the lithosphere (Thébault et al., 2010): at the large scale $O(1000)$ km, it overlaps with the core dynamo field; at the intermediate scale $O(100)$ km which is predominant, it can be measured by the high-precision geomagnetic satellites; and at the small scale $O(10)$ km, it cannot be accurately measured at the satellite altitude of about 500 km where the existing satellites usually orbit. The small-scale lithospheric magnetic field requires nearer surface measurements such as lower-altitude-perigee satellites together with ground, marine and airborne magnetic surveys. In other words, the new geomagnetic constellation is particularly suitable for the studies of intermediate or small scales of geomagnetic anomalies of the lithospheric origin.

Ground magnetic observatory and airborne magnetic survey conveniently and accurately measure magnetic fields of the Earth's lithosphere on its surface. At the present time, there exist more than 150 magnetic observatories distributed over the Earth's ground measuring geomagnetic fields with high accuracy < 0.1 nT. However, a severe inherent limitation of the observatory and airborne magnetic data is imposed by its geographical distribution that cannot cover the whole Earth to provide a wide spatial spectrum of the lithospheric magnetic field. Consequently, an accurate global magnetic surrey of the lithosphere requires a coordinated geomagnetic satellite constellation together with ground magnetic observatories. Moreover, it should be noticed that a huge amount of geomagnetic data from ground, marine and airborne magnetic surveys, because of its enormous economic values, are inaccessible to the general public and scientific community.

Mapping and understanding the magnetic signatures of the Earth's lithosphere is of huge challenge for several reasons below. (i) For large length-scales $O(1000)$ km, the core dynamo field masks the lithospheric field whose primary properties remain largely unknown. (ii) The raw magnetic field data from the near-Earth-surface measurements for constructing a lithospheric magnetic model must be corrected from the magnetic fields that are produced in the core dynamo, the mantle and the oceans, and the ionosphere and the magnetosphere—which are not fully understood or whose processes cannot be accurately modeled. Constructing a global lithospheric magnetic model without being affected by other magnetic sources remains a nearly impossible task. (iii) The Runcorn theorem (Runcorn, 1975) states that a constant magnetic susceptibility in a spherical shell does not produce visible external magnetic field signatures for an inducing dipole field internal to the shell. This suggests that there exists a large null space of the relevant inverse problem (Backus, 1970). Of course this non-uniqueness can be reduced by assuming that the lithospheric magnetic field is primarily produced by the induced magnetization together with prior known properties of the lithospheric geology. But there are no simple relationships between the lithospheric magnetic field and the geological structure. (iv) The lithospheric magnetic field is characterized by many sharp discontinuities and, consequently, it is numerically difficult to use the standard spherical harmonic expansion to model it.

At least two different mathematical approaches have been adopted to model the lithospheric magnetic field: the source dipoles formulation (Meyer et al., 1983) and the VIM (Vertically Integrated Magnetization) formulation (Gubbins et al., 2011). Below is a brief discussion of the VIM approach following Gubbins et al. (2011) and Gubbins et al. (2022), which is able to explicitly demonstrate the non-uniqueness of lithospheric inverse problems. Denote the lithospheric magnetization $\boldsymbol{M}(\boldsymbol{r})$ (either induced or remanent) by assuming that it is confined in the thin lithosphere of thickness d with $0 < d/r_{\mathrm{E}} \ll 1$. When $r > r_{\mathrm{E}}$, the magnetic potential $\Phi_M(\boldsymbol{r})$ produced by the magnetization $\boldsymbol{M}(\boldsymbol{r})$ satisfies Poisson's equation and can be computed by

$$\Phi_M(\boldsymbol{r}) = \frac{\mu}{4\pi}\int_{\tilde{V}}\left[\boldsymbol{M}(\tilde{\boldsymbol{r}})\cdot\tilde{\nabla}\left(\frac{1}{\boldsymbol{r}-\tilde{\boldsymbol{r}}}\right)\right]\mathrm{d}\tilde{V}, \tag{43}$$

where the volume integration $\int_{\tilde{V}}\mathrm{d}\tilde{V}$ is over the lithosphere with $\boldsymbol{M}(\boldsymbol{r})$. In the exterior of the Earth $r > r_{\mathrm{E}}$, the magnetic potential $\Phi_e(\boldsymbol{r})$ satisfying Laplace's equation $\nabla^2\Phi_e = 0$ has the solution

$$\Phi_e(\boldsymbol{r}) = r_{\mathrm{E}}\sum_{l,m}\left(\frac{r_{\mathrm{E}}}{r}\right)^{l+1}P_l^m\left(\tilde{g}_l^m\cos m\phi + \tilde{h}_l^m\sin m\phi\right), \tag{44}$$

where $P_l^m(\cos\theta)$ denotes a Schmidt-normalized associated Legendre function, and $(\tilde{g}_l^m, \tilde{h}_l^m)$ are the geomagnetic coefficients that are directly related to the lithospheric magnetization $\boldsymbol{M}(\boldsymbol{r})$. Upon equating (43) to (44), expanding $1/(\boldsymbol{r}-\tilde{\boldsymbol{r}})$ in Equation (43) in terms of the Legendre function, and carrying out the relevant integration, $(\tilde{g}_l^m, \tilde{h}_l^m)$ can be expressible as

$$\tilde{g}_l^m = \tilde{g}_l^m(\boldsymbol{M});\quad \tilde{h}_l^m = \tilde{h}_l^m(\boldsymbol{M}).$$

A key question is then how the geomagnetic coefficients $(\tilde{g}_l^m, \tilde{h}_l^m)$ representing an observable potential field is explicitly related to the structure and amplitude of the magnetization $\boldsymbol{M}(\boldsymbol{r})$. An answer to this question can be illustrated through the following four elements.

First, notice that there exist three different types of vector spherical harmonics (Blatt and Weisskopf, 1952; Gubbins et al., 2011) given by

$$\boldsymbol{Y}_{l,l+1}^{m}(\theta,\phi)=\frac{r^{l+2}}{\sqrt{(l+1)(2l+1)}}\nabla\left[\frac{1}{r^{l+1}}Y_l^m(\theta,\phi)\right], \tag{45}$$

$$\boldsymbol{Y}_{l,l}^{m}(\theta,\phi)=\frac{i}{\sqrt{l(l+1)}}\boldsymbol{r}\times\nabla\left[\frac{1}{r^{l+1}}Y_l^m(\theta,\phi)\right], \tag{46}$$

$$\boldsymbol{Y}_{l,l-1}^{m}(\theta,\phi)=\frac{1}{r^{l-1}\sqrt{l(2l+1)}}\nabla\left[r^{l}Y_l^m(\theta,\phi)\right], \tag{47}$$

where $Y_l^m(\theta,\phi)$ denotes complex spherical harmonics. It is well known that vector spherical harmonics given by Equations (45)–(47) are mathematically complete and orthogonal when integrated over a sphere.

Second, since the magnetization $\boldsymbol{M}(\theta,\phi,r)$ is confined in a thin lithosphere, the VIM vector $\boldsymbol{M}_0(\theta,\phi)$ can be introduced by averaging over the radial direction,

$$\boldsymbol{M}_0(\theta,\phi)\sim\int_{(r_E-d)}^{r_E}\boldsymbol{M}(\theta,\phi,r)\,\mathrm{d}r,$$

where d is constant and $0<d/r_E\ll 1$, to substantially simplify the mathematical analysis.

Third, because of the mathematical completeness of vector spherical harmonics Equations (45)–(47), any magnetization $\boldsymbol{M}_0(\theta,\phi)$ can be always expanded in the form

$$\boldsymbol{M}_0(\theta,\phi)=\sum_{l,m}\left(E_l^m\boldsymbol{Y}_{l,l+1}^m+I_l^m\boldsymbol{Y}_{l,l-1}^m+T_l^m\boldsymbol{Y}_{l,l}^m\right), \tag{48}$$

where E_l^m, I_l^m, T_l^m are complex coefficients of the expansion representing three different subsets, in which T_l^m is obviously a toroidal part of the magnetization $\boldsymbol{M}_0$. Furthermore, it can be shown (see Gubbins et al. (2011) for details) that

$$\tilde{g}_l^m(\boldsymbol{M}_0)\sim\mathrm{Real}\left[I_l^m\right];\quad \tilde{h}_l^m(\boldsymbol{M}_0)\sim\mathrm{Imag}\left[I_l^m\right].$$

In other words, the two other subsets, (E_l^m, T_l^m), of the magnetization $\boldsymbol{M}_0$ cannot produce a potential field measurable in the exterior of Earth $r>r_E$: if the magnetization component $I_l^m=0$ in Equation (48), the lithospheric magnetic field external to the Earth will be zero regardless of the size of other two components (E_l^m, T_l^m).

Fourth, assume that the core dynamo is confined in $0<r<r_o$ and that the core dynamo field $\boldsymbol{B}$ in $r_o<r<r_E$ is a potential field $\boldsymbol{B}=-\nabla\Phi_e$ with $\nabla^2\Phi_e=0$ and Φ_e in the complex form,

$$\Phi_e(\theta,\phi,r)=r_E\sum_{l,m}c_l^m\left(\frac{r_E}{r}\right)^{l+1}Y_l^m(\theta,\phi).$$

The inducing magnetic field $\boldsymbol{B}$ can be, then, expressed as

$$\begin{aligned}\boldsymbol{B}&\sim-\nabla\left[\sum_{l,m}c_l^m\left(\frac{r_E}{r}\right)^{l+1}Y_l^m(\theta,\phi)\right]\\&\sim\sum_{l,m}c_l^m\left[(l+1)Y_l^m,\frac{\partial Y_l^m}{\partial\theta},\frac{\partial Y_l^m}{\partial\phi}\right]\\&\sim\sum_{l,m}\left[c_l^m\boldsymbol{Y}_{l,l+1}^m(\theta,\phi)\right],\end{aligned} \tag{49}$$

where c_l^m denotes the coefficients of the expansion. If the top Earth's crust is a uniformly thin shell with a constant susceptibility χ magnetized by the core dynamo field $\boldsymbol{B}$ ($\boldsymbol{M}_0=\chi\boldsymbol{B}$), we then have

$$\boldsymbol{M}_0\sim\boldsymbol{B}\sim\sum_{l,m}\left[c_l^m\boldsymbol{Y}_{l,l+1}^m(\theta,\phi)\right], \tag{50}$$

which represents the E_l^m subset of the magnetization $\boldsymbol{M}_0$. It follows that, if the core dynamo generates only a dipole field, *i.e.*, $c_l^m=0$ except for $c_1^0\neq 0$, we recover Runcorn's theorem. In general, any structure of the magnetization $\boldsymbol{M}_0$ described by Equation (50) is unobservable in the exterior of the Earth $r>r_E$, representing a fundamental non-uniqueness of the inverse problem. When the Earth's lithosphere is non-uniform, *i.e.*, a variable susceptibility $\chi(\theta,\phi)$ or a variable thickness $d(\theta,\phi)$, Runcorn's theorem and its extension, because of the coupling of different components, no longer strictly apply, although the solution of a uniformly shell magnetized by an internal dynamo field may still represent a first-order approximation with a large part of the magnetization $\boldsymbol{M}$ being unable to produce observable magnetic fields.

In summary, the magnetization $\boldsymbol{M}(\theta,\phi,r)$ confined in a thin lithospheric layer is comprised of the three subsets in expansion (48): while the subset I_l^m produces an observable potential field in the exterior of the Earth, the subsets E_l^m and T_l^m are invisible (Gubbins et al., 2011). Magnetic anomalies of the Earth's lithosphere have to be modelled or interpreted, alike gravity and seismic anomalies, alongside geological and geophysical characteristics of the lithosphere. Over large regions of the Earth, there are no ground or marine or airborne magnetic measurements and, consequently, synthetic magnetic data need to be employed in those regions. In order to reduce the non-uniqueness, described by Runcorn's theorem and its extension, in constructing a lithospheric magnetic model, we may have to assume that the lithospheric magnetic field is purely produced by induced magnetization which, together with prior known properties of the lithospheric geology, constrains the direction and amplitude of the lithospheric field (Meyer et al., 1983). In other words, a better understanding of the crustal chemical composition, crustal thickness, crustal age, crustal rifting and variations of the core dynamo magnetic field helps construct a faithful lithospheric magnetic field model.

3.5 Unveiling Oceans Flow via Magnetic Signals

It is of significance to notice that the Earth's oceans are made of an electrically conducting fluid with relatively high conductivity (an average conductivity about 3.0 S/m) and that the large-scale salty water moving through the core dynamo field $\boldsymbol{B}$ can induce electric currents (Sanford, 1971), which generate the induced magnetic fields $\boldsymbol{B}_\mathrm{i}$ measurable by modern high-precision geomagnetic satellites (Tyler et al.,2003; Maus and Kuvshinov, 2004; Sabaka et al., 2015; Irrgang et al., 2017). The high-precision geomagnetic satellites like Swarm are capable of identifying magnetic signatures less than $O(1)$ nT at satellite altitude about 500 km or less, representing a really remarkable achievement of the modern geomagnetic satellites (Tyler et al., 2003). There exist two different components in the oceans-flow-induced magnetic field: an invisible toroidal field whose amplitude is about $O(100)$ nT and a visible poloidal field whose amplitude is about $O(10)$ nT in oceans.

Since the oceans-flow-induced magnetic signals are sensitive to the oceans bottom structure and conductance, the signals of the induced magnetic field can be employed to probe the geochemical and geological structure below the oceans floor. Extracting contributions from the the oceans-flow-induced magnetic field is also crucial to isolating the magnetic field that originates from the core dynamo. It is now well-established that the gravitational tidal flow in oceans can induce weak but measurable magnetic signals in the form of fast periodic magnetic perturbations, in contrast to the slow and chaotic secular variations of the core dynamo field. Previous studies have mainly focused on the oceans-induced magnetic field generated by the well-known lunar semidiurnal tide, M_2, with a period of about 12.4 hours (Tyler et al., 2003). This discrete frequency in connection with the motion of Sun and Moon helps distinguish the induced field from other sources such as the core dynamo field and the lithospheric field.

Magnetohydrodynamic equations describing the oceans-induced magnetic field via the tidal flow $\boldsymbol{u}_{\text{ocean}}$, in contrast to the core dynamo problem, can be substantially simplified. We first decompose the total magnetic field in oceans into the two components, (the core dynamo field $\boldsymbol{B}$ and the oceans induced field $\boldsymbol{B}_{\text{i}}$) and, then, assume that both the Lorentz force,

$$\frac{1}{\rho_{\text{ocean}}\mu}\left[\nabla\times(\boldsymbol{B}+\boldsymbol{B}_{\text{i}})\times(\boldsymbol{B}+\boldsymbol{B}_{\text{i}})\right],$$

where ρ_{ocean} denotes the density of electrically conducting water in oceans, and the viscous force, $\nu\nabla^2\boldsymbol{u}_{\text{ocean}}$, are negligibly small. The above assumptions lead to the following equation of motion for oceans:

$$\begin{aligned}\frac{\partial\boldsymbol{u}_{\text{ocean}}}{\partial t}&+\boldsymbol{u}_{\text{ocean}}\cdot\nabla\boldsymbol{u}_{\text{ocean}}+2\boldsymbol{\Omega}\times\boldsymbol{u}_{\text{ocean}}\\&=-\frac{1}{\rho_{\text{ocean}}}\nabla p_{\text{ocean}}+\boldsymbol{f}_{\text{tide}}(\boldsymbol{r},t),\end{aligned}\tag{51}$$

$$0=\nabla\cdot\boldsymbol{u}_{\text{ocean}},\tag{52}$$

where $\boldsymbol{f}_{\text{tide}}(\boldsymbol{r},t)$ represents the tidal forces that are periodic. In other words, the structure and amplitude of the oceans flow $\boldsymbol{u}_{\text{ocean}}$ are assumed to be totally decoupled from the magnetic effect in electrically conducting oceans, which dramatically simplifies the mathematical analysis. In principle, Equations (51)–(52) can be solved for a given tidal force $\boldsymbol{f}_{\text{tide}}(\boldsymbol{r},t)=\boldsymbol{f}_{\text{tide}}(\boldsymbol{r})\mathrm{e}^{\mathrm{i}\omega t}$, subject to a set of proper boundary conditions. In practice, we may simply use an ocean flow $\boldsymbol{u}_{\text{ocean}}(\boldsymbol{r},t)$ derived from an existing ocean model to further simplify the magnetic induction problem by avoiding solving Equations (51)–(52) (Stephenson and Bryan, 1992).

If the flow motion $\boldsymbol{u}_{\text{ocean}}$ is dynamically decoupled from the magnetic effects, the induction problem becomes purely kinematic and can be described by the following Ampere's law and Faraday's law in the magnetohydrodynamic limit,

$$\begin{aligned}&\frac{1}{\mu\sigma_{\text{ocean}}}\nabla\times[(\boldsymbol{B}(\boldsymbol{r},t)+\boldsymbol{B}_{\text{i}}(\boldsymbol{r},t)]\\&\quad=\{\boldsymbol{E}(\boldsymbol{r},t)+\boldsymbol{u}_{\text{ocean}}(\boldsymbol{r},t)\times[\boldsymbol{B}(\boldsymbol{r},t)+\boldsymbol{B}_{\text{i}}(\boldsymbol{r},t)]\},\end{aligned}\tag{53}$$

$$\frac{\partial\left[\boldsymbol{B}(\boldsymbol{r},t)+\boldsymbol{B}_{\text{i}}(\boldsymbol{r},t)\right]}{\partial t}=-\nabla\times\boldsymbol{E}(\boldsymbol{r},t).\tag{54}$$

A simplification for Equations (53)–(54) can be made by further assuming that (i) the core dynamo currents cannot leak into the mantle and oceans

$$\nabla\times\boldsymbol{B}(\boldsymbol{r},t)=0,$$

(ii) the core dynamo field is predominant with

$$|\boldsymbol{B}(\boldsymbol{r},t)|\gg|\boldsymbol{B}_{\text{i}}(\boldsymbol{r},t)|,$$

and (iii) the induced field $\boldsymbol{B}_{\text{i}}(\boldsymbol{r},t)$) varies much faster than the core dynamo field $\boldsymbol{B}(\boldsymbol{r},t)$)

$$\left|\frac{\partial\boldsymbol{B}_{\text{i}}(\boldsymbol{r},t)}{\partial t}\right|\gg\left|\frac{\partial\boldsymbol{B}(\boldsymbol{r},t)}{\partial t}\right|.$$

They give rise to the simplified equations

$$\frac{\nabla\times\boldsymbol{B}_{\text{i}}(\boldsymbol{r},t)}{\mu\sigma_{\text{ocean}}}=[\boldsymbol{E}(\boldsymbol{r},t)+\boldsymbol{u}_{\text{ocean}}(\boldsymbol{r},t)\times\boldsymbol{B}(\boldsymbol{r})],\tag{55}$$

$$\frac{\partial\boldsymbol{B}_{\text{i}}(\boldsymbol{r},t)}{\partial t}=-\nabla\times\boldsymbol{E}(\boldsymbol{r},t).\tag{56}$$

Upon adopting a harmonic time-dependence of the flow $\boldsymbol{u}_{\text{ocean}}$, the induced magnetic field $\boldsymbol{B}_{\text{i}}$ and the electric field $\boldsymbol{E}$ in the form

$$\begin{aligned}\boldsymbol{u}_{\text{ocean}}(\boldsymbol{r},t)&=\boldsymbol{u}_{\text{ocean}}(r,\theta,\phi)\mathrm{e}^{\mathrm{i}\omega t},\\\boldsymbol{B}_{\text{i}}(\boldsymbol{r},t)&=\boldsymbol{B}_{\text{i}}(r,\theta,\phi)\mathrm{e}^{\mathrm{i}\,\omega t},\\\boldsymbol{E}(\boldsymbol{r},t)&=\boldsymbol{E}(r,\theta,\phi)\mathrm{e}^{\mathrm{i}\omega t},\end{aligned}$$

we can combine Equations (55) and (56) into a single complex inhomogeneous Helmholtz equation

$$\begin{aligned}&\frac{1}{\mu\sigma_{\text{ocean}}}\nabla^2\boldsymbol{B}_{\text{i}}(r,\theta,\phi)-\mathrm{i}\omega\boldsymbol{B}_{\text{i}}(r,\theta,\phi)\\&\quad=-\nabla\times[\boldsymbol{u}_{\text{ocean}}(r,\theta,\phi)\times\boldsymbol{B}(r,\theta,\phi)],\end{aligned}\tag{57}$$

where σ_{ocean} is assumed to be constant with the right-hand side of the inhomogeneous Helmholtz equation being prescribed, representing a source term driving the induced magnetic field $\boldsymbol{B}_{\text{i}}$.

Above oceans in the atmosphere $r>r_{\text{E}}$, the magnetic field $\boldsymbol{B}_e$ is approximately the gradient of magnetic scalar potential Φ_e satisfying the Laplace's equation

$$\nabla^2\Phi_e(r,\theta,\phi)=0,\tag{58}$$

with the external potential field given by

$$\boldsymbol{B}_e=-\nabla\Phi_e.$$

Beneath oceans, the induced magnetic field $\boldsymbol{B}_{\text{m}}$ is governed by the homogeneous Helmholtz equation,

$$\nabla\times\left[\frac{1}{\mu\sigma_{\text{m}}(r,\theta,\phi)}\nabla\times\boldsymbol{B}_{\text{m}}(r,\theta,\phi)\right]+\mathrm{i}\omega\boldsymbol{B}_{\text{m}}(r,\theta,\phi)=0,\tag{59}$$

for a prescribed conductivity profile $\sigma_{\text{m}}(r,\theta,\phi)$.

It follows that three different regions must be considered in determining the induced magnetic field $\boldsymbol{B}_{\text{i}}$, a solution of the boundary-value problem characterized by complicated matching conditions between the three regions. For oceans in the domain $r_{\text{bottom}}(\theta,\phi)\le r\le r_{\text{E}}$, where r_{bottom} denotes the bottom boundary of oceans, we solve the inhomogeneous Helmholtz Equation (57) for a given flow $\boldsymbol{u}_{\text{ocean}}$ and a given core dynamo field $\boldsymbol{B}$. For atmosphere in the domain $r_{\text{E}}\le r\le\infty$, Equation (58) is solved subject to the conditions

$$-\nabla\Phi_e=\boldsymbol{B}_{\text{i}},\quad\boldsymbol{r}\cdot\nabla\times\boldsymbol{B}_{\text{i}}=0\ \text{ at }\ r=r_{\text{E}}\tag{60}$$

and the boundary condition

$$\nabla \Phi_e = 0 \ \text{at} \ r/r_{\mathrm{E}} \gg 1. \tag{61}$$

For the mantle in the domain $r_o \le r \le r_{\mathrm{bottom}}(\theta, \phi)$, we seek for a solution of Equation (59) subject to the condition at the core–mantle boundary

$$\boldsymbol{B}_{\mathrm{m}}(r_{\mathrm{i}}, \theta, \phi) = 0 \ \text{at} \ r = r_{\mathrm{o}}.$$

At the interface $r = r_{\mathrm{bottom}}(\theta, \phi)$ between the oceans and the mantle, the matching conditions that all components of the magnetic field and the tangential components of the electrical field are continuous give rise to

$$\boldsymbol{B}_{\mathrm{i}} = \boldsymbol{B}_{\mathrm{m}},$$

$$\boldsymbol{n}_{\mathrm{bottom}} \times \left(\frac{1}{\sigma_{\mathrm{ocean}}} \nabla \times \boldsymbol{B}_{\mathrm{i}} \right) = \boldsymbol{n}_{\mathrm{bottom}} \times \left(\frac{1}{\sigma_{\mathrm{m}}} \nabla \times \boldsymbol{B}_{\mathrm{m}} \right),$$

where $\boldsymbol{n}_{\mathrm{bottom}}$ denotes the normal of the interface $r_{\mathrm{bottom}}(\theta, \phi)$. This boundary condition may be extremely complicated because of the geometry of $r_{\mathrm{bottom}}(\theta, \phi)$ and the difference between σ_{ocean} and $\sigma_{\mathrm{m}}(r, \theta, \phi)$. Since $0 < (r_{\mathrm{E}} - r_{\mathrm{bottom}})/r_{\mathrm{E}} \ll 1$, a thin-shell approximation is usually adopted to simplify the mathematical formulation.

In the forward problem, we make use of a given tidal flow $\boldsymbol{u}_{\mathrm{ocean}}$, a given profile of the electric conductivity $\sigma_{\mathrm{m}}(r, \theta, \phi)$ and a given geometry of the oceans $r_{\mathrm{bottom}}(\theta, \phi)$, and a given core dynamo field $\boldsymbol{B}$ to predict the induced magnetic field $\boldsymbol{B}_{\mathrm{i}}$. In the inverse problem, we estimate the oceans flow $\boldsymbol{u}_{\mathrm{ocean}}$ by solving an optimization problem that minimizes a penalty function involving the induced magnetic field $\boldsymbol{B}_{\mathrm{i}}$ measured by high-precision geomagnetic satellites, the data misfit (between the observed field and the prediction of the forward problem), the parameterized tidal flow, the forward problem, and the regularization parameter and the regularization term. Finally, it is worth mentioning that the induced magnetic field measured by Galileo spacecraft was successfully employed to predict the existence of a global ocean in Europa from its response to the Jupiters magnetic field (Khurana et al., 2011).

3.6 Understanding Earth's External Magnetic Environment

While the Earth's neutral atmosphere comprises its troposphere extending from the ground surface to about 10 km coupled with its stratosphere extending from about 10 km to 60 km, the Earth's external magnetic environment mainly comprises its ionosphere (altitude approximately from 60 km to 200 km) dynamically coupled with its magnetosphere (altitude > 200 km) of the ionized plasma region (one of the four fundamental states of matter that is made of gas of ions and free electrons) with finite electric conductivity (Kivelson and Russell, 1995; Stern, 1996). The solar wind represents an expansion of the solar corona, which is composed mainly of electrons and ions, into interplanetary media in the solar system. At a very high speed of 250–1000 km/s, it also manifests a continuous loss of mass from the Sun, delivering approximately 5×10^{-2} W/m^2 of power to the Earth's magnetosphere (whose average radius is about $15r_{\mathrm{E}}$) with its total power input of approximately 10^{15} W. Note that the total energy consumption on the Earth is approximately 10^{13} W. The magnetopause—where the magnetic pressure resulting from the core dynamo field balances the dynamic pressure due to the solar wind located at about $10r_{\mathrm{E}}$ along the Sun–Earth line—marks the outer boundary of the Earth's magnetosphere. Across the magnetopause, the solar wind transfers its energy into the magnetosphere, drives the highly active dynamics and results in the complex electric currents that can be indirectly measured by high-precision geomagnetic satellites.

Through depositing high-energy charged particles into the magnetosphere primarily in the Earth's polar regions, the solar wind provides electromagnetic and kinetic energies to both the magnetosphere and the ionosphere, where Alfven waves may play a significant role in coupling the magnetosphere and the ionosphere. In the Earth's space magnetic environment, the Van Allen radiation belts—which include an inner zone with mainly high-energy protons extending from just above the neutral atmosphere to about 10,000 km and an outer zone with mainly high-energy electrons extending to about $6r_{\mathrm{E}}$ at low and middle latitudes—represent an important toroidal region of the Earth's magnetically trapped charged particles. It is the geomagnetic field that controls and shapes the change, location and structure of charged particles in the Van Allen belts (Kivelson and Russell, 1995).

Sustained primarily by continuous interaction of the core dynamo magnetic field with the solar wind, there exist complex electric current systems in the ionosphere and the magnetosphere which are rapidly changing (Ganushkina et al., 2018). The field-aligned current is generated by the electrons and ions movement along the geomagnetic field lines, connecting the Earth's high-latitude ionosphere (the auroral zones) with the magnetosphere (Siscoe et al., 1991). The ring current is maintained by the azimuthal drift of charged particles (10–200 KeV) that are trapped on the geomagnetic field lines in the region $2–6r_{\mathrm{E}}$ around the Earth's equatorial plane whose strength may be indicated by the *Dst* index calculated using the magnetic measurements of a network of ground-based stations (Ganushkina et al., 2018). Intensive injections of high-speed solar-wind particles, along with ionospheric ions, into the Earth's magnetosphere can cause the rapid ring-current intensification and lead to global-scale geomagnetic storms. The Solar Quiet current in the ionospheric region is mainly confined between approximately 80 and 200 km heights, referred to as the ionospheric dynamo (Richmond, 1979), which is mainly driven by the solar tidal winds and manifests itself as regular variations on the Earth's ground magnetic observations. It should be highlighted that the electric current systems in the Earth's ionosphere and magnetosphere are highly nonlinear, complicated and not fully understood. For the illustrative purpose, below is a brief description of the Solar Quiet current taking place in the ionosphere. Note that the key physics of the ionospheric dynamo is quite different from the core dynamo: in the electrically conducting ionosphere, the movement of plasma $\boldsymbol{u}_{\mathrm{i}}$ across the existing core dynamo magnetic field $\boldsymbol{B}$ yields an electromotive force $\boldsymbol{u}_{\mathrm{i}} \times \boldsymbol{B}$ that produces an electric field $\boldsymbol{E}_{\mathrm{i}}$ and an electric current $\boldsymbol{J}_{\mathrm{i}}$ (Stewart, 1882).

The ionosphere is comprised of a partially ionized gas lying between the atmosphere and the magnetosphere, where Sun's X-ray and extreme ultraviolet radiation, as well as the highly energetic electrons and protons, continually separate the neutral air molecules into ions and electrons. In comparison to the nearly uniform electric conductivity for the core dynamo, the electric conductivity of the ionospheric dynamo is highly anisotropic (*i.e.*, described by a tensor) because both the direction and amplitude of the magnetic field $\boldsymbol{B}$ and the electric field $\boldsymbol{E}$ control the movement of ions and electrons in the ionosphere. For instance, there exist the parallel conductivity parallel to $\boldsymbol{B}$, the Pedersen conductivity parallel to $\boldsymbol{E}$ but orthogonal to $\boldsymbol{B}$, and the Hall conductivity orthogonal to both $\boldsymbol{B}$ and $\boldsymbol{E}$. The anisotropic property of the electric conductivity in the ionosphere—whose tensor depends on geomagnetic activities, the time of day and seasons—dramatically complicates its magnetohydrodynamics. But the basic dynamics of the ionosphere may be roughly illustrated by introducing a thin-layer approximation with its thickness Δh and the height-integrated conductivity σ_i, which is approximately described by the linear horizonal momentum equation (Volland, 1984),

$$\frac{\partial \boldsymbol{u}_{\rm i}}{\partial t} + 2\Omega\hat{\boldsymbol{r}} \times \boldsymbol{u}_{\rm i}\cos\theta = -\frac{1}{\rho_{\rm i}}\overline{\nabla}p_{\rm i} + \frac{1}{\rho_{\rm i}\Delta h}\boldsymbol{J}_{\rm i} \times \boldsymbol{B}, \tag{62}$$

where $\boldsymbol{u}_{\rm i}(\theta,\phi,t)$ denotes the horizonal ionospheric tidal flow, $\overline{\nabla}p_{\rm i}(\theta,\phi,t)$ is the horizonal pressure gradient and $\boldsymbol{J}_{\rm i}(\theta,\phi,t)\times\boldsymbol{B}(\theta,\phi)$ represents the horizonal Lorentz force with the core dynamo field $\boldsymbol{B}$ being given and $\rho_{\rm i}$ is the background density of the ionosphere.

Mathematically, the linear homogeneous Equation (62), together with other linear equations (such as the linearized continuity equation and Ohm's law), form an eigenvalue problem whose eigenfunctions are usually referred to as the solar tidal wave modes. Physically, Sun's differential thermal radiation causes night-day pressure and temperature differences, which result in solar tidal waves in the Earth's mantle frame of reference with a basic period of about one solar day and force the movement of the ionospheric electrically conducting air against the background geomagnetic field $\boldsymbol{B}$. In a simple description, the above physical effects excite or resonate with a particular set of the solar tidal waves mode,

$$(p_{\rm i}, \boldsymbol{J}_{\rm i}, \boldsymbol{u}_{\rm i})\,(\theta,\phi,t) \sim f_n^m(\theta){\rm e}^{{\rm i}m(\phi+\omega_{\rm i}t)},$$

which satisfies the equation of motion (62) and other relevant linearized equations, where m is the azimuthal wave number, n is the latitudinal wave number, and $\omega_{\rm i}$ denotes the frequency of a tidal wave mode. The most fundamental mode resonating with the solar tidal effect is the solar diurnal ($m = 1, n = -2$) tidal wave mode whose approximate solution (Volland, 1984) is given by

$$p_{\rm i}(\theta,\phi,t) \sim (1 + 4\cos^2\theta) \sim \theta{\rm e}^{{\rm i}(\phi+t)} \tag{63}$$

and

$$\boldsymbol{J}_{\rm i}(\theta,\phi,t) \sim \nabla\times\left[\hat{\boldsymbol{r}}\,(\sigma_{\rm i}\sin\theta\cos\theta)\,{\rm e}^{{\rm i}(\phi+t)}\right], \tag{64}$$

where

$$(\sigma_{\rm i}\sin\theta\cos\theta){\rm e}^{{\rm i}(\phi+t)} = \text{constant},$$

describes the approximate streamlines of the Solar Quiet electric current in the ionosphere, which is antisymmetric with respect to the equator if $\sigma_{\rm i}$ is constant.

The Earth's magnetic field measured by the low-Earth-orbit satellites such as Macau Science Satellite-1 (MSS-1) is dominated by the component generated by the core geodynamo—the core dynamo field at mid-latitudes is typically about 45,000 nT while the external transient field of the ionospheric and magnetospheric origin is typically about 200 nT. However, it is the external-origin magnetic field that represents a major obstacle in accurately resolving temporal and spatial variations of the core dynamo field. Various electric currents in the ionosphere and magnetosphere make a significant contribution to the measured magnetic fields in space and, hence, it is vital that an accurate geomagnetic field model provides a suitable treatment of the external sources that are rapidly changing. An accurate measurement of the Earth's highly variable magnetic field on a wide range of spatial–temporal scales resulting from the ionospheric and magnetospheric processes is essential to understanding the Earth's external magnetic environment, particularly when the prevailing solar–terrestrial interaction leads to the development of a strong geomagnetic storm (Stolle et al., 2016).

It should be underlined that the magnetic measurements provided by the highly-eccentric-orbit-satellite constellation are able to sample the three-dimensional (200–5000 km) structure of the upper ionosphere and the lower magnetosphere and, hence, help construct a three-dimensional model to understand the ionospheric and magnetospheric dynamic processes. The new constellation also offers a capacity of investigating the process of the magnetosphere–ionosphere coupling process. Several nearly-circular-orbits geomagnetic satellites (such as Swarms and Champs) have measured the properties of the ionosphere and the magnetosphere along their circular orbits and, consequently, can only reveal in situ two-dimensional structure. Since the highly elliptic orbits of the new constellation cover an extended range of the three-dimensional space of the upper ionosphere and lower magnetosphere, it will offer a better opportunity of deriving empirical models for ionospheric and magnetospheric dynamic processes and constructing an accurate global geomagnetic field model with a higher temporal–spatial resolution.

3.7 Constructing High-Resolution Global Geomagnetic Models

On the basis of magnetic measurements on the Earth's ground surface, Jackson et al. (2000) produced a global geomagnetic model represented by a time-dependent (spanning 400 years from 1590 to 1990) spherical harmonic expansion up to degree and order 14. In the satellite era, on the basis of combined surface and satellite magnetic measurements, Finlay et al. (2017), for example, produced a CHAOS-6 global geomagnetic model spanning from 1999 to 2017 with a higher global spatial resolution, clearly revealing changes of the geomagnetic field in various temporal–spatial scales. In particular, it unveils a rapid change of the South Atlantic Anomaly (SAA) which may be caused by a reverse magnetic flux patch at the Earth's fluid core beneath Southern Africa. It represents a special region where charged particles in the innermost Van Allen belt can reach the altitude of Earth's upper atmosphere. It should be especially noticed that, since both surface and satellite magnetic measurements are limited only at some particular time or space, providing the direction and amplitude of a geomagnetic field at any time and space in the exterior of the Earth requires constructing a global geomagnetic model $\boldsymbol{B}_{\rm model}(r,\theta,\phi,t)$ with $r \ge r_{\rm E}$. A high-precision and high-resolution geomagnetic model $\boldsymbol{B}_{\rm model}(r, \theta, \phi, t)$ can be usually validated by the measurements of such as ground observatories and geomagnetic satellites.

The observed geomagnetic field consists of contributions from various sources that need to be disentangled, representing a highly challenging task in constructing an accurate global geomagnetic model $\boldsymbol{B}_{\rm model}(r, \theta, \phi, t)$. In general, $\boldsymbol{B}_{\rm model}(r, \theta, \phi, t)$ may be expressed as

$$\begin{aligned}\boldsymbol{B}_{\rm model} = {} & \boldsymbol{B}_{\rm core} + \boldsymbol{B}_{\rm mantle} \\ & + \boldsymbol{B}_{\rm crust} + \boldsymbol{B}_{\rm ocean} + \boldsymbol{B}_{\rm iono} + \boldsymbol{B}_{\rm magneto}\cdots,\end{aligned}$$

where $\boldsymbol{B}_{\rm core}$ is the core dynamo field (slow varying), $\boldsymbol{B}_{\rm crust}$ is the lithospheric field (temporally constant), $\boldsymbol{B}_{\rm mantle}$ denotes the magnetic field induced in the mantle and lithosphere (fast varying), $\boldsymbol{B}_{\rm ocean}$ denotes the oceans induced field (fast varying), $\boldsymbol{B}_{\rm iono}$ is associated with the ionospheric current systems (fast varying), and $\boldsymbol{B}_{\rm magneto}$ originates from the magnetospheric current systems (fast varying). Note that (i) our knowledge of the core dynamo field $\boldsymbol{B}_{\rm core}$, whose toroidal component is invisible, is primarily restricted to its poloidal component measured at the Earth's surface and (ii) in comparison with the amplitude of $\boldsymbol{B}_{\rm core}$, even the largest contribution from external sources, such as $\boldsymbol{B}_{\rm crust}$ and $\boldsymbol{B}_{\rm iono}$, is relatively small at or near the Earth's surface.

When a satellite trajectory is away from magnetic and current sources, the Earth's magnetic field $\boldsymbol{B}_{\rm model}(r, \theta, \phi, t)$ can be derived from a potential $V_{\rm E}(r, \theta, \phi, t)$ defined as

$$\boldsymbol{B}_{\rm model} = -\nabla V_{\rm E}(r, \theta, \phi, t),$$

which, together with the solenoidal condition $\nabla \cdot \boldsymbol{B}_{\rm model} = 0$, gives rise to Laplace's equation

$$\nabla^2 V_{\rm E}(r, \theta, \phi, t) = 0. \tag{65}$$

It is important to notice that (i) the vector $\boldsymbol{B}_{\rm model}$ can be conveniently written in a local north, east and center coordinate system, (ii) an assumption (current-source free) leads to Laplace's Equation (65) and (iii) its simple solution breaks down in the auroral and ionospheric regions where satellite trajectories may cross strong electrical currents. In other words, break-down regions cannot be properly modeled in a mathematical formulation that leads to Laplace's Equation (65) while errors related to the un-modeled processes may dominate the error budget in certain regions of a global geomagnetic model. A solution to Laplace's Equation (65) in spherical geometry can be simply written in terms of spherical harmonics expansion

$$\begin{aligned}V_{\rm E} = {} & r_{\rm E}\sum_{l,m}\left(\frac{r_{\rm E}}{r}\right)^{l+1}\left(g_l^m\cos m\phi + h_l^m \sin m\phi\right)_{\rm in} P_l^m \\ & + \left(\frac{r}{r_{\rm E}}\right)^{l}\left(q_l^m\cos m\phi + s_l^m \sin m\phi\right)_{\rm ex} P_l^m,\end{aligned} \tag{66}$$

where the first term represents the internal sources (relative to satellite measurements, such as the core dynamo and lithospheric magnetization), the second term is for the external sources (relative to satellite measurements, such as magnetospheric currents), $g_l^m(t)$ and $h_l^m(t)$ are the Gauss coefficients related to the internal sources (such as those being originated from the core dynamo and mantle induction), $q_l^m(t)$ and $s_l^m(t)$ are the Gauss coefficients related to the external processes such as magnetospheric currents which are usually parameterized, $P_l^m(\cos\theta)$ is the Schmidt normalized associated Legendre function with l denoting its degree and m for its order. A high-precision magnetic vector field measured by a satellite constellation represents the superposition of all contributions from various sources and, consequently, it is a massive challenge, alongside a huge benefit, that we are capable of making the sophisticated separation of different sources and determining the spatial−temporal structure of the sources.

Electric currents in the ionosphere < 200 km are mathematically regarded as an internal field because they are underneath the usual satellite altitude 200−450 km. In consequence, one cannot, based on the solution of Equation (65) using satellite data, to readily separate the ionosphere−dynamo field from the core−dynamo field without imposing extra constraints. Lithospheric fields are regarded as being internal to both observatory and satellite measurements. It should be highlighted that separating different-sources signatures of satellite magnetic measurements requires a subtle and complicated geomagnetic modeling that involves many physical and mathematical processes alongside many modeling parameters. In Equation (66), the Gauss coefficients $(g_l^m, h_l^m, q_l^m, s_l^m)$, which represent the geomagnetic model parameters, are usually derived by making a least-square fit to the magnetic field measurements at the satellite altitude, to be discussed briefly below. However, the least-square fit is mathematically formulated as an inverse problem that is ill-posed, underdetermined. In other words, the magnetic measurements at particular time and space are not sufficient to determine all the Gauss coefficients $(g_l^m, h_l^m, q_l^m, s_l^m)$, representing a well-known non-uniqueness property of the inverse problem.

There exist mainly two different approaches in global geomagnetic field modelings that determine the Gauss coefficients in Equation (66). The first is fully comprehensive, which attempts to include all the major magnetic contributions (the core dynamo, lithospheric, ionospheric, magnetospheric, the mantle/crust induced, the oceans induced) by including the magnetic field data at all local times at all latitudes together with the parameterisations of all different sources and the simultaneous inversion of all parameters (Sabaka et al., 2004; 2015). The modeling technique and procedure in this comprehensive approach are complicated. The second is highly restrictive, focusing mainly on the large-scale and slowly-changing components (the core dynamo, lithosphere and magnetospheric ring currents) of the Earth's magnetic field (Finlay et al., 2015; 2017). Below is a brief description of the simpler second approach of global geomagnetic field modelings for the purpose of illustration.

When the trajectory of a geomagnetic satellite is at about 450 km from the Earth's surface, its measurements of the geomagnetic field are far away from the magnetic sources which lie in the Earth's liquid core and lithosphere (the internal sources), and in the magnetosphere (the external source). In this simple approach, separating different magnetic signatures of the satellite measurements is relatively straightforward (Finlay et al., 2017). The following three steps are taken in the global geomagnetic field modelings:

(i) Making use of the magnetic field data only obtained in the geomagnetically quiet regions and times (defined by a small *Dst*-

index or a small *Kp*-index) by selecting only the night-side data in order to reduce the magnetic contamination due to the ionospheric diurnal currents, by selecting only the middle-low latitude data or giving the polar region data with less weights in order to reduce the influence of the polar ionospheric currents, and by selecting only the solar quiet data in order to separate internal and external contributions. It should be noted that different selection rules and different prior knowledge imposed actually lead to different global geomagnetic models and that the geomagnetic data are usually weighted by $\sin\theta$ to reflect the equal-area distribution of spherical geometry.

(ii) Rotating the magnetic field vector $\boldsymbol{B}_{\rm obs}$ measured in the vector fluxgate magnetometer to the vector $\boldsymbol{B}_{\rm model} = -\nabla V_{\rm E}$ in the mantle frame of coordinate systems through

$$\boldsymbol{B}_{\rm obs} = -\mathcal{M}_1 \cdot \mathcal{M}_2 \cdot \mathcal{M}_3 \cdot \nabla V_{\rm E}\left(r, \theta, \phi, g_l^m, h_l^m, q_l^m, s_l^m\right), \tag{67}$$

where $\mathcal{M}_3$ denotes a matrix that rotates the field vector from the mantle frame to the International Celestial Reference Frame based on the information of the satellite position and time; the $\mathcal{M}_2$ is a matrix that rotates the field vector from the International Celestial Reference Frame to the star camera frame based on the attitude data observed by the star camera fixed on the optical bench; and $\mathcal{M}_1$ is a matrix that rotates the field vector from the star camera frame to the orthogonal magnetometer frame via the three Euler angles which can be co-estimated as the model parameters. Equation (67) relates the observed field $\boldsymbol{B}_{\rm obs}$ in the magnetometer frame to the parameters of the geomagnetic field model $(g_l^m, h_l^m, q_l^m, s_l^m)$ along with the three Euler angles. A magnetic field vector measured by a vector magnetometer is usually re-scaled such that its scalar size is in agreement with the measurement of a scalar magnetometer on the satellite.

(iii) Minimizing, via a regularized least-squares algorithm which is typically nonlinear and thus iterative, the cost function

$$\begin{aligned}&(\boldsymbol{d}_{\rm obs} - \boldsymbol{d}_{\rm mod})^{\rm T}\, \overline{C}_{\rm cm}\, (\boldsymbol{d}_{\rm obs} - \boldsymbol{d}_{\rm mod}) \\ &\quad + \left[\frac{\lambda}{\Delta t}\int_0^{\Delta t}\int_0^{2\pi}\int_0^{\pi}\left(\frac{\partial^2 \boldsymbol{B}_{\rm model}}{\partial t^2}\right)^2 \sin\theta\, {\rm d}\theta\, {\rm d}\phi\right],\end{aligned} \tag{68}$$

where $\boldsymbol{d}_{\rm obs}$ denotes the data vector of the observed magnetic field in the magnetometer frame, $\boldsymbol{d}_{\rm mod}$ represents the model prediction in the magnetometer frame, $\overline{C}_{\rm cm}$ is the data covariance matrix which may be weighed, and λ is a parameter that controls the degree of the regularization during the time span of the model. In the cost function (68), the squared value of the second time derivative of the magnetic field over the Earth's surface is penalized in the regularization. It should be pointed out, however, that there are no theories that favor a particular regularization and, for example, the cost function may take the form

$$\begin{aligned}&(\boldsymbol{d}_{\rm obs} - \boldsymbol{d}_{\rm mod})^{\rm T}\, \overline{C}_{\rm cm}\, (\boldsymbol{d}_{\rm obs} - \boldsymbol{d}_{\rm mod}) \\ &\quad + \left[\frac{\lambda}{\Delta t}\int_0^{\Delta t}\int_0^{2\pi}\int_0^{\pi}\left(\frac{\partial^3 \boldsymbol{B}_{\rm model}}{\partial t^3}\right)^2 \sin\theta\, {\rm d}\theta\, {\rm d}\phi\right],\end{aligned} \tag{69}$$

or take a combination of both the second and third time derivatives or other forms. There are also no theories that favor a particular value of the regularization (or damping) parameter λ which is usually selected by carrying out numerical experiments such as comparing with the measurements of geomagnetic observatories. Some prior information on the physical nature of the problem, for example, the maximum heat flux at the core–mantle boundary, may be also desirable as prior constraints and regularization conditions. Of course, the minimization of the cost function can also take place in other frames such as the Earth's mantle frame. It is of primary importance to notice that a geomagnetic field model that fits the measurements is typically non-unique and that even perfect measurements fail to produce a unique geomagnetic model.

With the new geomagnetic constellation and its unprecedentedly accurate measurements, a higher-precision, higher-resolution and fully comprehensive geomagnetic model is expected to be constructed by an integration of satellite and ground-based magnetic data, facilitated by a careful sampling of the temporal and spatial data to disentangle different magnetic sources. Designing an innovative algorithm that is capable of reducing the non-uniqueness of the inverse problem in a geophysically reasonable way also represents an essential key to deriving a higher-precision and higher-resolution geomagnetic field model.

3.8 Exploring Applications of High-Precision Global Geomagnetic Field

The capacity of the new geomagnetic survey constellation enables higher-dimensional (in comparison to the previous geomagnetic survey satellites) magnetic measurements to be made, offering the mapping of a high-precision global magnetic field $\boldsymbol{B}(\boldsymbol{r}, t)$ and its secular variation $\partial\boldsymbol{B}(\boldsymbol{r}, t)/\partial t$. In addition to scientific applications previously discussed, it also has a wide range of practical applications with huge benefits to modern human societies. Below is a list of several possible practical applications.

(i) **Global Navigation in Modern Technology**: The geomagnetic field has been traditionally employed as means of global navigation in aerospace and sailing, while its applications recently become ubiquitous in modern technological societies including smart phones, laptops and tablets (Goldenberg, 2006). It should be noticed that GNSS (Global Navigation Satellite Systems) uses the geomagnetic field in its navigation system to determine heading. There are many factors that can affect certainty and accuracy of the GNSS navigation system: one significant error is the estimated orientation of moving progression in the GNSS positioning computation via the orientation of the Earth's magnetic field. The GNSS navigation is also marked by high uncertainties in some special circumstances and does not work in underwater or underground. It is therefore highly desirable to develop the capability of reliable global navigation by using the high-precision geomagnetic field map as an alternative that is not only relatively accurate but also, comparing to a network of vulnerable GNSS satellites, extremely difficult to jam and interfere. In fact, global navigation based on the variation of the geomagnetic field mapped to geographic locations has been growing in popularity in the community of global navigation. The geomagnetic information, combined with other sensors such as an integrated inertial and magnetometer navigation system, can produce a more reliable global navigation solution. A magnetometer measuring variations of the geomagnetic field and then comparing to an existing geomagnetic field map together with a suitable navigation algorithm may be deployed as the sole source of position and orientation information for reliable global navigation.

(ii) **Natural Resource Exploration**: There exist many distinguished magnetic features in connection with the properties of magnetized rocks in the crust and upper mantle. Oil and gas exploration industries strongly depend on the high-precision geomagnetic data which allow to easily survey a large-area geological structure and to search for a special region that may have potential economic values. The accurate charts of lithospheric magnetic field are often employed to interpret the magnetic anomalies, similar to gravity anomalies and seismological anomalies, as special geological and geophysical properties of the crust in facilitating resource explorations. Extracting the lithospheric magnetic signatures—which are much weaker compared to the core dynamo field—requires a highly resolved model of the Earth's lithospheric magnetic field produced by an integration of both satellite and ground magnetic data. It is of hopes that the new geomagnetic constellation offers a greatly improved chart of the lithospheric magnetic field with finer resolution and higher accuracy and help identify the hidden geological structure and natural resources.

(iii) **Mitigating Geomagnetic Effects on Ground Technological Infrastructures**: On a timescale of days, Coronal Mass Ejection can eject a huge amount of solar material into interplanetary space and frequently result in strong geomagnetic storms. The largest geomagnetic storms can produce large time-dependent magnetic disturbances up to several thousands (> 1000) nT at the Earth's ground surface. According to Faraday's law of induction, $\partial \boldsymbol{B}/\partial t = -\nabla \times \boldsymbol{E}$, a large time derivative of the magnetic disturbances can drive a strong electric field on the Earth's ground and, hence, induce a strong electric current (which is usually referred to as geomagnetically induced currents) flowing in large-scaled electrically grounded conductors—such as power grids, rail ways and pipe lines—of modern ground-based technological systems. Geomagnetically induced currents can suddenly flow into a power-grid-transmission network (which is the electrically low-resistance path on the Earth), disrupt the supply of electric power that is connected to the Earth's ground, cause serious damage and even lead to a blackout (Guillon et al., 2016). Since the frequency of geomagnetically induced currents is much lower than the 50 Hz usually adopted in the modern power grids, they can drive dangerous vibrations, produce heat dissipations in the transformers and result in catastrophic consequences. Through the electric ground connection of modern railways system, geomagnetically induced currents can also disrupt the electrically powered high-speed train (Liu LG et al., 2016) and affect the rail track circuits and their normal function with "wrong-side failures" (the signals turn to green when they should be red, which may cause the danger of train clashes). It is of hopes that the new constellation with an unprecedented precision offers an improved understanding and forecasting of variations of the geomagnetic field and helps mitigate the impact of geomagnetically induced currents on modern large-scale technological infrastructure on the Earth's ground.

(iv) **Mitigating Geomagnetic Effects on Space Technological Infrastructures**: The geomagnetic field largely controls the Earth's space environment and its rapid spatial–temporal variations can also result in the damage of modern space infrastructures such as the global navigation systems operated by GNSS and the normal operation of satellites. Signals received from at least 4 satellites of a GNSS constellation at altitude about 20,000 km provide a position for a GNSS receiver which converts the travel times into distances and, then, computes its geographic location. In the computation, the GNSS signals—which are delayed when passing through the geomagnetically-controlled ionosphere and, hence, may give rise to inaccuracies—must be corrected. A geomagnetic storm can dramatically alter the total electron content on the GNSS signal path such that its effects cannot be fully and accurately corrected. In this circumstance, GNSS users may have to switch to an alternative navigation system during a strong geomagnetic storm. Moreover, the geomagnetic field also controls the size of the density gradient in the ionospheric plasma which may cause instabilities of the Rayleigh–Taylor type, leading to the ionospheric scintillation which modulates the phase and amplitude of the GNSS signals such that they become practically un-useable. As a consequence of the electric currents and Joule dissipation in the lower ionosphere during a geomagnetic storm, the upper neutral atmosphere can be quickly heated up and rise buoyantly to significantly enhanced drag on low-altitude satellites around 300–500 km, leading to the troubles of identifying and tracking a satellite as well as earlier re-entries of a satellite. There exist a wide range of modern space technological infrastructures, ranging from navigation systems to satellite operations, which are critically affected by the Earth's external space environment that is largely controlled by the Earth's magnetic field (Luntama, 2017).

4. Concluding Remarks

It is the Sun's dynamo operating in its deep interior and generating the solar magnetic field that controls all observed major eruptive phenomena on the Sun. Solar flares and coronal mass ejections likely result from magnetic-field instabilities (for example, Zhang K and Busse, 1995) that suddenly release magnetic energy stored in the complex magnetic structures in the solar atmosphere and convection zone, representing the primary sources of high-speed particles causing disturbances in the Earth's external magnetic environment. In other words, the Earth's external magnetic activities are closely correlated with the solar 11-year dynamo cycle, which is defined as the interval successive maximum in sunspot numbers. The year of 1755 began the first solar cycle in the contemporary accounting; the 24th solar cycle marked by much less solar eruptive activities was ended around the year of 2020; and the new geomagnetic constellation discussed in this review is timely to offer an important service in the forthcoming 25th solar cycle.

The geomagnetic temporal spectrum measurable by satellites and observatories is widely broad, encompassing all timescales from seconds to decades while the corresponding spatial spectrum is also widely broad, ranging from the size of the Earth originating from the core dynamo to the very small scale due to the magnetized crust rocks. It should be noted that the International Geomagnetic Reference Field (IGRF) represents a series of digital models of the Earth's main magnetic field and its secular variations, which are contributed by many groups of volunteer scientists in different counties with different weights. It should be also noted that an IGRF model is marked by low spatial–temporal resolutions with both large length scales (larger than 1000 km) and long time scales (longer than one year). A higher spatial–temporal resolution model of the geomagnetic field requires a fully comprehensive approach by including all the significant magnetic sources: the core dynamo, lithosphere, ionosphere, magnetosphere, mantle and oceans. Airborne and marine magnetic surveys are typically with short length scales ranging from a few hundred meters to a few hundred kilometers. It is of hopes that lower-altitude satellite measurements characterized by the highly eccentric orbits of perigee about 200 km offer a critical overlap between two (airborne/marine and satellites) different data sets of the geomagnetic field. An important goal of the new geomagnetic constellation is, —by making use of the integrated marine, airborne, observatory and satellite magnetic data together with an innovative algorithm, —to construct a new generation of the global geomagnetic model with an unprecedented high spatial-temporal resolution. Achieving this goal represents a difficult challenge because the relevant inverse problem is ill-posed, under-determined and non-unique.

The new constellation of geomagnetic survey satellites expects to make a significant contribution to mapping the Earth's magnetic fields with finer resolution and higher accuracy, revealing more detailed information about the Earth's core dynamo, constructing three-dimensional profiles of the Earth's mantle conductivity, enabling three-dimensional observations of the upper ionosphere and lower magnetosphere via in situ measurements of the geomagnetic fields, understanding the dynamic state of Sun–Earth interaction and Earth's external magnetic environment, developing highly accurate models of the Earth's magnetic field along with the discoveries of new natural phenomena and the verification of new geomagnetic theories.

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